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Problems of a cut in a three-dimensional elastic wedge☆

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Abstract

The Ritz variational method is applied to problems of a crack (a cut) in the middle half-plane of a three-dimensional elastic wedge. The faces of the elastic wedge are either stress-free (Problem A) or are under conditions of sliding or rigid clamping (Problems B and C respectively). The crack is open and is under a specified normal load. Each of the problems reduces to an operator integrodifferential equation in relation to the jump in normal displacement in the crack area. The method selected makes it possible to calculate the stress intensity factor at a relatively small distance from the edge of the wedge to the cut area. Numerical and asymptotic solutions [Pozharskii DA. An elliptical crack in an elastic three-dimensional wedge. *Izv. Ross Akad. Nauk. MTT* 1993;(6):105–12] for an elliptical crack are compared. In the second part of the paper the case of a cut reaching the edge of the wedge at one point is considered. This models a V-shaped crack whose apex has reached the edge of the wedge, giving a new singular point in the solution of boundary-value problems for equations of elastic equilibrium. The asymptotic form of the normal displacements and stress in the vicinity of the crack tip is investigated. Here, the method employed in [Babeshko VA, Glushkov YeV, Zinchenko ZhF. *The dynamics of Inhomogeneous Linearly Elastic Media.* Moscow: Nauka; 1989] and [Glushkov YeV, Glushkova NV. Singularities of the elastic stress field in the vicinity of the tip of a V-shaped three-dimensional crack. *Izv. Ross Akad. Nauk. MTT* 1992;(4):82–6] to find the operator spectrum is refined. The new basis function system selected enables the elements of an infinite-dimensional matrix to be expressed as converging series. The asymptotic form of the normal stress outside a V-shaped cut, which is identical with the asymptotic form of the contact pressure in the contact problem for an elastic wedge of half the aperture angle, is determined, when the contact area supplements the cut area up to the face of the wedge.

In an earlier paper,[1] the regular asymptotic method was used, which is effective when there is a sufficient distance between the crack and edge. The Ritz method was employed in Ref. 4 to solve the analogous problem of a crack in an elastic layer. A narrow crack emerging at the edge of a wedge was examined in Ref. 5 using the paired integral equation method. The problem of a crack close to the free boundary of an elastic half-space[6,7] is a special case of the problem being solved here (a wedge angle of 180°). The method employed in the second part of the paper is effective whatever the aperture angles of the crack and the elastic wedge and was applied earlier to contact problems with a V-shaped contact area.[8,9] A regular asymptotic solution for a V-shaped crack with a small aperture angle in an elastic wedge showed[10] that there are no oscillating terms associated with complex points of the spectrum. Singularities in the case of a crack reaching a regular surface have been investigated,[11] and also singularities of the elastic stresses at the vertex of a polyhedron.[12]

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1. The Ritz method for an elliptical crack in a wedge

The formulation of Problems A, B and C of a cut in the middle half-plane of a wedge with three different types of boundary condition on the wedge faces was given earlier in Section 1 of Ref. 1. These problems reduce to an integrodifferential equation (Eq. (1.2) of reference 1). Below, we will retain the notation adopted earlier in Ref. 1, but with two misprints corrected: 1) in the formula for $g_2(t)$ (formula (1.4) in Ref. 1), cth αt rather than cth2αt should have been printed; 2) in the formula for $g_*(u)$ (formula (1.10) in Ref. 1), 2cos2α rather than cos2α should have been printed in the numerator.

Let the region of the crack be an ellipse Ω extended along the edge of the wedge:

$$(r-a)^2/c^2+z^2/b^2\le 1,\quad b\ge c$$

We will introduce dimensionless quantities by formulae (formulae (2.1) in Ref. 1) containing two principal geometrical parameters: $\lambda=a/b$ and $c'=c/b$ (the primes will be omitted below). For simplicity we will assume that a constant normal load q is applied to the sides of the cut. We then have (Eq. (1.2) in Ref. 1) the following integrodifferential equation in the function $f(r, z)$ of the cut opening

$$-\Delta_{rz}\iint\limits_{\Omega}\frac{f(x,y)}{R}dxdy+\iint\limits_{\Omega}f(x,y)F(x,y,r,z)dxdy = 2\pi q,\quad (r,z)\in\Omega \tag{1.1}$$

where $R=((r-x)^2+(z-y)^2)^{1/2}$ and the function $F(x, y, r, z)$ was defined earlier (see formula (1.8) in Ref. 1 for Problem A or formula (1.11) in Ref. 1 for Problems B and C, with x replaced by $x+\lambda$ and r by $r+\lambda$).

In the limiting case $\lambda\to\infty$, Eq. (1.1) has the exact solution

$$f(r,z) = A\sqrt{s(r,z)},\quad A = \frac{cq}{\mathrm{E}(e)},\quad e = \sqrt{1-c^2},\quad s(r,z) = 1-\frac{r^2}{c^2}-z^2 \tag{1.2}$$

corresponding to a crack in an unbounded elastic medium [5]. Here, E(e) is the complete elliptical integral of the second kind.

Assuming that $\lambda>c$ (the cut does not reach the edge of the wedge), to solve Eq. (1.1) we will use the Ritz variational method. We will investigate the minimum of the functional

$$\mathcal{G}(f) = (f,\mathcal{B}f)-4\pi(f,q)$$

where $\mathcal{B}$ is the operator on the left-hand side of Eq. (1.1). We will seek a solution in the form

$$f(r,z) = A\sqrt{s(r,z)}\sum_{m=0}^{M}\sum_{n=0}^{N}\left[A_{mn}U_{2m}\left(\frac{r}{c}\right)+B_{mn}U_{2m+1}\left(\frac{r}{c}\right)\right]\cos(\pi nz) \tag{1.3}$$

where $U_m(x)$ are Chebyshev polynomials of the second kind.

We will use the expansion (formula 2.5.6.4 in Ref. 13, and formula 2.16.14.1 in Ref. 14)

$$-\Delta_{rz}\frac{1}{R} = \frac{2}{\pi}\int\limits_0^{\infty}\int\limits_0^{\infty}\sqrt{u^2+v^2}\cos(u(r-x))\cos(v(z-y))dudv \tag{1.4}$$

understood in the generalized sense. Introducing representation (1.4) into Eq. (1.1), on the basis of the conditions for a minimum of the functional $\mathcal{G}$ we obtain a system of linear algebraic equations in the unknown coefficients a_{mn} and B_{mn} ($m=0, 1, \ldots, M$; $n=0, 1, \ldots, N$). Here, the modified Bessel function must be replaced by its integral representation (formula 2.4.18.4 of in Ref. 13) and account must be taken of the values of the integrals (formula 2.5.44.13 in Ref. 13, and formula 2.12.4.6 in Ref. 14)

$$\begin{aligned}
&\iint_\Omega \cos Ax \cos By \sqrt{s(x, y)}\,dxdy = 2\pi c F_0(cA, B)\\
&F_0(s, t) = \frac{\sin H - H\cos H}{H^3}, \quad H = \sqrt{s^2 + t^2}\\
&\iint_\Omega U_{2m}\left(\frac{x}{c}\right)\cos Ax \cos By \sqrt{s(x, y)}\,dxdy = 2\pi c F_{2m}(cA, B)\\
&\iint_\Omega U_{2m+1}\left(\frac{x}{c}\right)\sin Ax \cos By \sqrt{s(x, y)}\,dxdy = 2\pi c F_{2m+1}(cA, B)
\end{aligned} \tag{1.5}$$

The last two integrals of (1.5) are evaluated by differentiating the first formula of (1.5) with respect to A. The first few functions $F_m(s, t)$ have the form

$$\begin{aligned}
&F_1(s, t) = 2s\frac{(3 - H^2)\sin H - 3H\cos H}{H^5}\\
&F_2(s, t) = \frac{(12 - 5H^2)\sin H - (12H - H^3)\cos H}{H^5} - 4s^2\frac{(15 - 6H^2)\sin H - (15H - H^3)\cos H}{H^7}
\end{aligned} \tag{1.6}$$

The system of linear equations is written as follows:

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{M}\sum_{l=0}^{N} A_{kl}P_{2k, l, 2m, n} + \frac{c^2}{\pi}\sum_{k=0}^{M}\sum_{l=0}^{N}(A_{kl}P^1_{2k, l, 2m, n} + B_{kl}P^2_{2k+1, l, 2m, n}) = \pi \mathrm{E}(e)F_{2m}(0, \pi n)\\
&\frac{1}{2}\sum_{k=0}^{M}\sum_{l=0}^{N} B_{kl}P_{2k+1, l, 2m+1, n} + \frac{c^2}{\pi}\sum_{k=0}^{M}\sum_{l=0}^{N}(-B_{kl}P^1_{2k, l, 2m, n} + A_{kl}P^2_{2k, l, 2m+1, n}) = 0
\end{aligned} \tag{1.7}$$

where the elements that are independent of the aperture angle of the wedge 2α, i.e. those corresponding to a crack in an unbounded medium and associated with the principal part of the operator on the left-hand side of Eq. (1.1), have the form

$$\begin{aligned}
&P_{\kappa, l, \mu, n} = \int_0^\infty\int_0^\infty \sum F_\kappa(u, \upsilon \pm \pi l)\sum F_\mu(u, \upsilon \pm \pi n)\sqrt{u^2 + c^2\upsilon^2}\,dud\upsilon\\
&\kappa = 2k, 2k + 1; \quad \mu = 2m, 2m + 1
\end{aligned} \tag{1.8}$$

Here and below, we have introduced the following notation

$$\sum f(\pm a) = f(+a) + f(-a)$$

The remaining matrix components of system (1.7), to avoid unwieldiness, will be given for a Poisson's ratio $\nu = 1/2$:

$$P^1_{\kappa, l, \mu, m} = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty u^2[W_j^{-1}(u) - \mathrm{cth}\pi u]\,\mathrm{sh}\pi u \cos us \cos ut \times$$

$$\times \int_{\beta \mathrm{ch} t}^\infty \exp(-\lambda p_1)\sum F_\kappa(icp_1, \beta \pm \pi l)dp_1 \int_{\beta \mathrm{ch} s}^\infty \exp(-\lambda p_2)\sum F_\mu(icp_2, \beta \pm \pi n)dp_2 du d\beta ds dt$$

$$P^2_{\kappa, l, \mu, n} = iP^1_{\kappa, l, \mu, n} \tag{1.9}$$

$$W_1(u) = \frac{\mathrm{sh}2\alpha u + u\sin 2\alpha}{2(\mathrm{sh}^2\alpha u - u^2\sin^2\alpha)}, \quad W_2(u) = \frac{\mathrm{ch}2\alpha u - \cos 2\alpha}{\mathrm{sh}2\alpha u + u\sin 2\alpha}$$

$$W_3(u) = \frac{\mathrm{sh}2\alpha u - u\sin 2\alpha}{\mathrm{ch}2\alpha u + u^2(1 - \cos 2\alpha) + 1}$$

Here, $j = 1$, 2 and 3 for Problems A, B and C respectively.

In the general case, it is necessary to add to elements (1.9) components generated by terms at $(1 - 2\nu)$ in the kernels, and to take into account the dependence of the function $W_3(u)$ on ν (see formulae (1.4), (1.8), and (1.11) in Ref. 1).

For elements (1.9), the conditions of symmetry

$$P^1_{\kappa, l, \mu, n} = P^1_{\mu, n, \kappa, l}, \quad P^2_{\kappa, l, \mu, n} = P^2_{\mu, n, \kappa, l}$$

are satisfied, ensuring symmetry of the matrix of system of equations (1.7) and resulting from the symmetry of the kernel of Eq. (1.1).

On the basis of solution (1.3), we obtain a formula for the ratio of the normal stress intensity factors (SIFs) for the case considered and for the case of a crack in space ($\lambda = \infty$)

$$\frac{K_I}{K_I^\infty} = \sum_{m=0}^{M}\sum_{n=0}^{N}[A_{mn}U_{2m}(\cos\psi) + B_{mn}U_{2m+1}(\cos\psi)]\cos(\pi n\sin\psi) \tag{1.10}$$

where ψ is the angle between the positive direction of the semi-axis r and a ray stemming from the origin of coordinates in a direction towards the point on the crack contour at which the ratio (1.10) is calculated.

To test the Ritz method, we will use a regular asymptotic solution obtained earlier,[1] effective for large values of λ (far from the edge of the wedge). This will be written in the form

$$f(r, z) = A\sqrt{s(r, z)}\left\{1 - \frac{A_* c^2}{2\lambda^3}\left[\frac{2}{3\mathrm{E}(e)} - \frac{e^2 r}{\lambda \mathrm{D}(e)}\right] + O\left(\frac{1}{\lambda^5}\right)\right\}$$
$$\mathrm{D}(e) = (2 - c^2)\mathrm{E}(e) - c^2\mathrm{K}(e) \tag{1.11}$$

where $K(e)$ is the complete elliptic integral of the first kind.

The constant A_* when $\nu = 1/2$ is calculated by means of the formula

$$A_* = \int_0^\infty u^2[W_j^{-1}(u)\mathrm{th}\pi u - 1]du \tag{1.12}$$

On the basis of solution (1.11), the SIF ratio (1.10) acquires the form

$$\frac{K_I}{K_I^\infty} = 1 - \frac{A_* c^2}{2\lambda^3}\left[\frac{2}{3\mathrm{E}(e)} - \frac{e^2 c\cos\psi}{\lambda \mathrm{D}(e)}\right] + O\left(\frac{1}{\lambda^5}\right) \tag{1.13}$$

Table 1

2α	λ	1.5		1.0		0.8		0.7	
	c	0.5	0.6	0.5	0.6	0.5	0.6	0.5	0.6
	f_0	0.419	0.479	0.434	0.507	0.459	0.561	–	–
180°	K_1	1.02	1.02	1.04	1.05	1.08	1.11	–	–
	K_2	1.02	1.03	1.08	1.14	1.21	1.50	–	–
	f_0	–	–	0.414	0.472	0.416	0.476	0.418	0.480
270°	K_1	–	–	1.01	1.01	1.01	1.01	1.01	1.01
	K_2	–	–	1.01	1.01	1.02	1.04	1.03	1.09

where the angle ψ has the same meaning as in (1.10). Below, we will put $K_1 = K_I/K_I^\infty$ when $\psi = 0$, and $K_2 = K_I/K_I^\infty$ when $\psi = \pi$.

For Problem A with $2\alpha = 180°$ (a half-space) and $\nu = c = 1/2$ (here, the constant $A_* = -0.4375$) we will give the results of testing the Ritz method: for an opening of the cut at its centre $f_0 = f(0, 0)/q = 0.415$ (0.414), and the SIFs have the values $K_1 = 1.010$ (1.003) and $K_2 = 1.011$ (1.005). The asymptotic solution is given in brackets; for $2\alpha = 180°$ it is applicable when $\lambda > 1 + c$.[1]

The Ritz method can be used when $\lambda > c$ and gives acceptable results for non-acute wedge aperture angles 2α. When $2\alpha \le 90°$, difficulties arise with the accurate evaluation of the integrals with respect to u in formulae (1.9), since

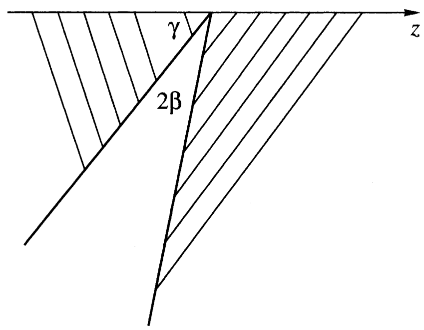
$$W_j^{-1}(u) - \operatorname{cth}\pi u = O(\exp(-2\alpha u)) \quad \text{when} \quad u \to +\infty$$

The Ritz method is particularly effective at angles $2\alpha \ge \pi$, when the wedge is transformed into an angular notch in elastic space (a V-shaped groove).

Table 1 gives values of f_0 and the SIFs for Problem A with $\nu = 1/2$ and different values of 2α, λ and c. The crack will first begin to propagate towards the edge, and therefore K_2 is always greater than or equal to K_1. On approaching the edge (λ decreases, c increases), the value of K_2 increases if the angle of the wedge decreases. The opening of the cut f_0 also increases as the region of the cut approaches the edge of the wedge. The smaller the angle of the wedge and the relative distance from the area Ω to the edge, the more dangerous is the crack in the sense of its subsequent propagation.

2. A V-shaped crack in a wedge

It is well known[6] that the normal stresses outside a V-shaped cut with angle 2β in elastic space at the cut tip have the same asymptotic form as the contact stresses in the problem of a V-shaped punch with angle 2π–2β on an elastic half-space. The problems of a cut with angle 2β in the middle half-plane of an elastic wedge with angle 2α and of the action of a punch on the face of an elastic wedge with angle α, when the contact area in plan occupies this entire face, with the exception of the V-shaped notch with angle 2β (this area is shown hatched in the figure), are related in a similar way.



Let the region of the cut Ω be a wedge with angle 2β (not hatched in the figure); the angle between the negative semi-axis z and the neighbouring side of the V-shaped cut is γ. In the initial integrodifferential equation (Eq. (1.2) in

Ref. 1), we will change to polar coordinates according to the formulae

$$x = \rho\cos\psi, \quad y = \rho\sin\psi, \quad r = r_*\cos\varphi, \quad z = r_*\sin\varphi$$

$$\Delta_{rz} \to \Delta_{r_*\varphi} = \frac{\partial^2}{\partial r_*^2} + \frac{1}{r_*}\frac{\partial}{\partial r_*} + \frac{1}{r_*^2}\frac{\partial^2}{\partial\varphi^2}; \quad f(x, y) = f_*(\rho, \psi), \quad q(r, z) = q_*(r_*, \varphi) \tag{2.1}$$

and seek its solution in the form of the Mellin integral

$$f_*(\rho, \psi) = \frac{1}{2\pi i}\int_\Gamma f_s^*(\psi)\rho^{-s-3/2}ds \tag{2.2}$$

where the contour Γ must be chosen from the condition $f_*(0, \psi) = 0$. We will use values of the known integrals (formula 2.16.14.3 in Ref. 14, and formula (1.7) in Ref. 15) and we will introduce the notation

$$R_\pm^u(s, \varphi) = \frac{1}{2}\Gamma\left(\frac{1}{2} + s + iu\right)\left\{\begin{matrix}\text{cosec}\\ \text{sec}\end{matrix}\right\}\left[\frac{\pi}{2}\left(\frac{1}{2} + s - iu\right)\right][P_{s-1/2}^{-iu}(\sin\varphi) \pm P_{s-1/2}^{-iu}(-\sin\varphi)] \tag{2.3}$$

where $\Gamma(z)$ is the gamma function and $P_s^{-iu}(x)$ is the Legendre function, to evaluate which it is convenient to use its representation (formula 8.704 in Ref. 16) in terms of a hypergeometric function.

If the right-hand side of the equation is now represented in the form

$$q_*(r_*, \varphi)\frac{1-\nu}{G} = \frac{1}{2\pi i}\int_\Gamma q_s^*(\varphi)r_*^{-s-5/2}ds \tag{2.4}$$

and the new notation

$$\varphi = \beta x + \beta_*, \quad \psi = \beta\xi + \beta_*, \quad \beta_* = \beta + \gamma - \pi/2, \quad f_s^*(\psi) = f_s(\xi), \quad \beta q_s^*(\varphi) = q_s(x)$$

is introduced, we will arrive at a one-dimensional integrodifferential equation of the form

$$-\left[\frac{d^2}{dx^2} + \beta^2\left(s + \frac{1}{2}\right)^2\right]\int_{-1}^{1} f_s(\xi)K_s(\beta(x-\xi))d\xi +$$

$$+\beta^2\int_{-1}^{1} f_s(\xi)F_s(\beta x + \beta_*, \beta\xi + \beta_*)d\xi = \pi q_s(x), \quad |x| \le 1$$

$$K_s(t) = \frac{\pi}{2\cos\pi s}P_{s-1/2}(-\cos t) \tag{2.5}$$

$$F_s(\varphi, \psi) = \frac{1}{2}\int_0^\infty \operatorname{sh}\pi u\left\{\frac{u^2(W_j^{-1}(u) - \operatorname{cth}\pi u)}{\cos\varphi\cos\psi}\sum[R_\pm^u(-s, \psi)R_\pm^u(s+1, \varphi)] -\right.$$

$$\left.-(1-2\nu)g_n(u)\sum[R_\pm^u(-s+1, \psi)R_\pm^u(s+2, \varphi)] + \tilde{F}_s(u, \varphi, \psi)\right\}du$$

For Problem A ($j=1$, $n=2$)

$$\tilde{F}_s(u,\varphi,\psi) = \frac{uW_*(u)}{2\cos\varphi\cos\psi\,\mathrm{ch}(\pi u/2)}\sum[\Phi_\pm(u)R^u_\pm(s+1,\varphi)] +$$
$$+(1-2\nu)g_*(u)\int_0^\infty \frac{1}{\mathrm{ch}\pi t+\mathrm{ch}\pi u}\left\{W_*(t)t\,\mathrm{sh}\pi t\sum[R^t_\pm(-s,\psi)R^u_\pm(s+2,\varphi)]-\right. \tag{2.6}$$
$$\left.-W_2(t)\,\mathrm{sh}\frac{\pi t}{2}\sum[\Phi_\pm(t)R^u_\pm(s+2,\varphi)]\right\}dt$$

where the functions $\Phi_\pm(t)$ are found from Fredholm equations of the second kind

$$\Phi_\pm(u)-(1-2\nu)\int_0^\infty L_2(u,y)\Phi_\pm(y)dy = X_\pm(u),\quad 0\le u<\infty$$

$$X_\pm(u) = 2(1-2\nu)\mathrm{ch}\frac{\pi u}{2}\int_0^\infty \frac{\mathrm{sh}\pi y g_*(y)}{\mathrm{ch}\pi u+\mathrm{ch}\pi y}R^y_\pm(-s+1,\psi)dy- \tag{2.7}$$
$$-2(1-2\nu)\int_0^\infty L_2(u,y)y\,\mathrm{ch}\frac{\pi y}{2}\frac{W_*(y)}{W_2(y)}R^y_\pm(-s,\psi)dy$$

For Problems B and C ($j=n=2$ and 3 respectively), $\tilde{F}_s(u,\varphi,\psi)=0$.

The index of singularity of the function $f_*(\rho,\psi)$ when $\rho\to 0$ is related to points of the spectrum of the operator on the left-hand side of Eq. (2.5). The poles s_k of the function $f_s(\xi)$ will be those values of the parameter s at which non-trivial solutions of the corresponding homogeneous equation may exist, i.e. points of the spectrum of the operator (2.5); here, s_k is independent of ξ.

To find s_k, Eq. (2.5) is discretized by Bubnov's method: the solution is sought in the form of an expansion in a system of basis function $\upsilon_m(\xi)$

$$f_s(\xi) = \sum_{m=0}^{\infty} t_m(s)\upsilon_m(\xi) \tag{2.8}$$

and, to determine $t_m(s)$, the error is analysed by the second basis $\{u_l\}_{l=0}^{\infty}$; as a result, in relation to the unknowns, the following infinite system arises

$$\sum_{n=0}^{\infty} a_{lm}t_m = q_l,\quad l = 0,1,\ldots \tag{2.9}$$

where

$$a_{lm} = -\frac{1}{\pi}\int_{-1}^{1}\left\{\left[\frac{d^2}{dx^2}+\beta^2\left(s+\frac{1}{2}\right)^2\right]\int_{-1}^{1}K_s(\beta(x-\xi))\upsilon_m(\xi)d\xi\right\}w_l(x)dx+$$
$$+\frac{\beta^2}{\pi}\int_{-1}^{1}\int_{-1}^{1}F_s(\beta x+\beta_*,\beta\xi+\beta_*)\upsilon_m(\xi)w_l(x)d\xi dx \tag{2.10}$$
$$q_l = \int_{-1}^{1}q_s(x)w_l(x)dx$$

Here, $\{w_k\}_{k=0}^{\infty}$ is a system of projectors onto the basis $\{u_l\}_{l=0}^{\infty}$, i.e. $(u_l, w_k)|_{L_2} = \delta_{kl}$, where δ_{kl} is the Kronecker delta.

The function $f_s(\xi)$ at $\xi=\pm 1$ behaves like $(1-\xi^2)^{1/2}$. To regularize the initial ill-posed problem – Eq. (2.5) – it is necessary to take into account this singularity in the coordinate functions, and we will therefore select the following as the basis functions (formula 7.343 in Ref. 16)

$$\upsilon_m(\xi) = \sqrt{1-\xi^2}U_m(\xi), \quad u_l(x) = 2U_l(x)$$

where $U_m(x)$ are Chebyshev polynomials of the second kind. In the system of functions $u_l(x)$ the singularity is not introduced since the right-hand side and the error are continuous functions. By virtue of the condition of orthogonality of the Chebyshev polynomials, we obtain that

$$w_k(x) = \sqrt{1-x^2}U_k(x)/\pi$$

Using an expansion of the Legendre function of the form[2,3]

$$\frac{1}{2\cos\pi s}P_{s-1/2}(-\cos(t-p)) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} G_k(s)\exp(ik(t-p))$$

$$G_k(s) = \frac{\tilde{G}_k(s)\tilde{G}_k(-s)}{2\tilde{G}_s(s+1)\tilde{G}_k(-s+1)}, \quad \tilde{G}_k(s) = \Gamma\left(\frac{s}{2}+\frac{|k|}{2}+\frac{1}{4}\right)$$

and also the value of the integral obtained using well-known formulae (formulae 7.355, 8.471 and 8.941 in Ref. 16)

$$\int_{-1}^{1}\exp(\pm iax)\sqrt{1-x^2}U_m(x)dx = \pi\exp\left(\pm i\frac{\pi m}{2}\right)\frac{m+1}{a}J_{m+1}(a)$$

twice differentiating with respect to x under the integration sign in formula (2.10) and, in system (2.9), making the replacements

$$a^*_{lm} = \frac{a_{lm}}{(l+1)(m+1)}, \quad t_m = \frac{t^*_m}{m+1}, \quad f^*_l = \frac{f_l}{l+1}$$

for the elements $a^*_{lm} = a^*_{lm}(s)$ $(l, m=0, 1, \ldots)$ we obtain

$$a^*_{lm}(s) = {\sum_{n=0}^{\infty}}' G_n(s)\frac{n^2-(s+1/2)^2}{\beta^2 n^2}J_{l+1}(\beta n)J_{m+1}(\beta n)\cos\left(\pi\frac{l-m}{2}\right)+ \\ +\frac{1}{\pi^2(l+1)(m+1)}\int_{-1}^{1}\int_{-1}^{1}F_s(\beta x+\beta_*, \beta y+\beta_*)\sqrt{1-x^2}\sqrt{1-y^2}U_m(x)U_l(y)dxdy \tag{2.11}$$

The prime on the summation sign denotes that the first term ($n=0$) of the series is taken with coefficient ½.

Owing to the successfully chosen system of basis functions, the series in formula (2.11) converges. If, when $s=s_k$, the determinant of the infinite-dimensional matrix with elements (2.11) vanishes, then, as follows from expression (2.2)

$$f_*(\rho, \psi) \sim \rho^{\gamma_*}, \quad 0<\gamma_*<1, \quad \gamma_* = -s_k-3/2 \text{ when } \rho\to 0$$

Table 2

Problem	2β	2α						
		45°	90°	135°	180°	225°	270°	315°
	45°	0.048	0.83	0.90	0.95	0.92	0.93	0.94
A	90°	0.96	0.97	0.75	0.81	0.78	0.79	0.79
	135°	0.94	0.72	0.66	0.65	0.65	0.65	0.64
	45°	0.90	0.88	0.90	0.96	0.93	0.93	0.94
B	90°	0.77	0.74	0.75	0.81	0.78	0.78	0.80
	135°	0.67	0.66	0.65	0.64	0.64	0.65	0.64
	45°	0.98	0.96	0.95	0.93	0.96	0.96	0.96
B	90°	0.89	0.85	0.81	0.79	0.81	0.82	0.82
	135°	0.72	0.67	0.66	0.65	0.64	0.64	0.64

Here, the normal stress outside the cut (the contact pressure in the problem of a punch in an additional region for a wedge with angle α), according to formula (2.4), has the asymptotic form

$$q_*(\rho, \psi) \sim \rho^{\gamma_* - 1} \quad (\rho \to 0)$$

For $\beta > 0.1\pi$ and not too low values of α, it is sufficient to truncate the matrix with elements (2.11) to a dimensional of 10–12 in order to ensure two significant digits for the zeros of its determinant on the real axis.

Problem B with $2\alpha = 360°$ and $F_s(\varphi, \psi) = 0$ corresponds to a crack in elastic space. Here we obtain $\gamma_* = 0.500$ for $2\beta = 180°$[7] and $\gamma_* = 0.815$ for $2\beta = 90°$;[17] at other crack angles, the index γ_* corresponds to the singularity $\gamma_* - 1$ of the contact pressures at the tip of a V-shaped punch with angle $2\pi - 2\beta$ on an elastic half-space.[8,9]

Table 2 gives values of γ_* for the three problems with $\beta_* = 0$ (the axis of symmetry of the cut is perpendicular to the edge of the wedge), $\nu = 0.3$, and different angles of the crack and the elastic wedge. The influence of the boundary conditions on the faces of the wedge on the asymptotic forms required falls as the aperture angle 2α of the elastic wedge increases. For low crack angles, calculations agree with results obtained earlier.[10] For example, with $2\alpha = 270°$, $2\beta = 0.4$ and $\beta_* = 0$ for the three problems, we have $0.98 < \gamma_* \leq 0.99$, which corresponds to a graph obtained earlier (Fig. 10 in Ref. 2).

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